Certified and fast computation of supremum norms of approximation errors

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Outline

Motivation

- Mathematical Libraries
- Correctly rounded elementary functions
- Supremum norm of error functions
- Previous approaches and difficulties

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- Safe and fast supremum norm of approximation errors
- Automatic differentiation
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Results & Conclusion

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- Problem: Most of them do not provide correctly rounded functions. This can ruin portability. IEEE-754 standard revision (June 2008) recommends correct rounding.
- Arenaire team develops the Correctly Rounded Libm (CRLibm)[1].

[1] http://lipforge.ens-lyon.fr/www/crlibm/

Mathematical Libraries (libms)

- Need: Correctly rounded functions
- Given $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$, the function code should always return the machine number closest to the exact value $f(x)$
- Use an approximation polynomial p of f over $[a, b]$, consider $\varepsilon = f - p \quad \text{or} \quad \varepsilon = \frac{p}{f} - 1$
- Given μ, basic condition to assure: $\forall x \in [a,\ b],\ |\varepsilon(x)| \leq \mu$

Supremum Norms of Error Functions

- Error $\varepsilon(x) = f(x) - p(x) \quad \text{or} \quad \varepsilon(x) = \frac{p(x)}{f(x)} - 1, \quad x \in [a, b]$
- Define $\|\varepsilon\|_\infty = \sup_{x \in [a, b]}\{|\varepsilon(x)|\}$
- Purpose: Compute a certified bound for the supremum norm of an error function
- Given p and f find an interval $\mathbf{r}$ (as thin as desired) such that $\|\varepsilon\|_\infty \in \mathbf{r}$.

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Need for a fast and certified algorithm:

- Correctly rounded elementary functions
- For computing the minimum error between a function and thousands of polynomials with floating-point coefficients

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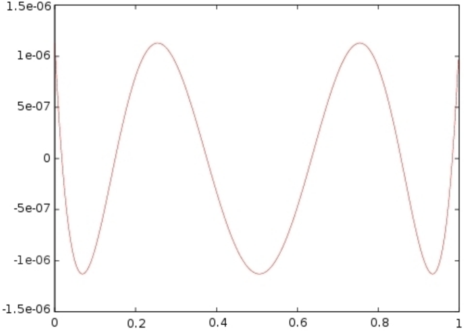
Evaluate using interval arithmetic —

$$F([-1,2]) = [-1,2]^2 + [-1,2] + [1,1]$$
$$F([-1,2]) = [0,4] + [-1,2] + [1,1]$$
$$F([-1,2]) = [0,7]$$

Motivation - Supremum norm of error functions

- Specific problem: supremum norm of error function
- Let $f(x) = e^x, x \in [-1, 1]$
- We are given p with real coefficients, $\deg p \leq 5$ s.t. $\|f - p\|_\infty$ is as small as possible (Remez algorithm)



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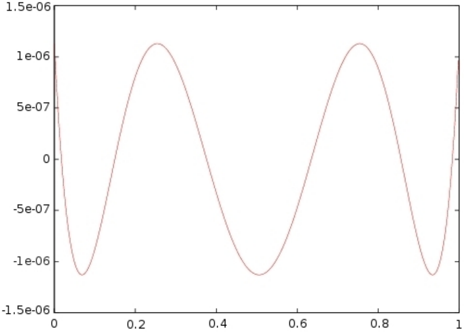

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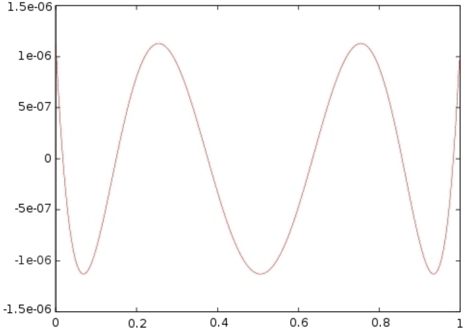

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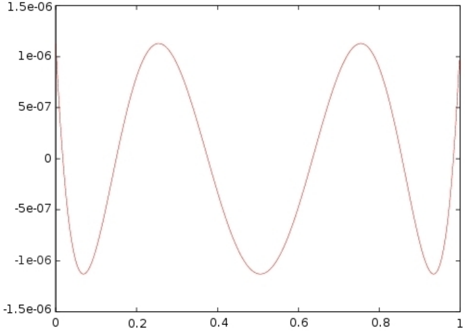

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  $x \in [-2, 2], f(x) = x - x,$ computing using interval extension,
  $F(X) = [-2, 2] - [-2, 2] = [-4, 4] \neq [0, 0]$
- Can be reduced by using very small intervals

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- Techniques based on a high order Taylor expansion of the error function and a sufficiently close bounding of the remainder
  - Sum of squares algorithms (Harrison)
  - The remainder is computed manually

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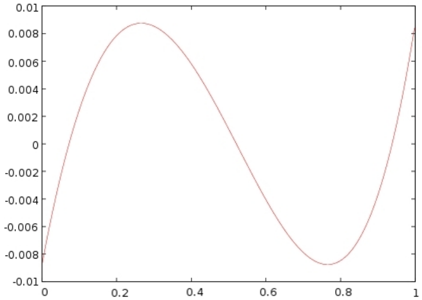
- By sampling we obtain: $\|f - p\|_\infty \simeq 0.008756064$



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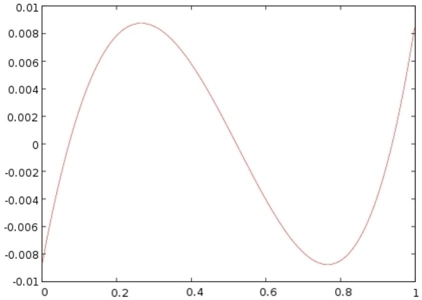

Certified bound of $\|f - p\|_\infty$

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$$\|f - p\|_\infty \leq \|f - T\|_\infty + \|T - p\|_\infty$$

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2. Automatic Differentiation
3. Bounding the polynomial difference - Roots isolation and refinement techniques

Our Approach - (1) Introducing a higher degree approximation polynomial

Let $n \in \mathbb{N}$, n times differentiable function f over $[a, b]$ around x_0:

$$f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(x_0)(x - x_0)^i}{i!} + \Delta_n(x, \xi)$$

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$\Delta_n(x, \xi) = \dfrac{f^{(n)}(\xi)(x - x_0)^n}{n!}$, $x \in [a, b]$, ξ lies strictly between x and x_0

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- Can be easily extended to work with interval arithmetic
- Compute $\frac{f^{(i)}(x_0)}{i!}, i = \{0, \cdots, n\}$ or an enclosure for $\frac{f^{(i)}(\xi)}{i!}, i = \{0, \cdots, n\}$, when $\xi \in [a, b]$.

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$$T(x) = \sum_{i=0}^{6} \frac{f^{(i)}(1/2)}{i!}(x - 1/2)^i = \sum_{i=0}^{6} \frac{\exp(1/2)}{i!}(x - 1/2)^i$$

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- Compute an enclosure of the remainder: Use AD

$$\Delta_7(x, \xi) = \underbrace{\frac{f^{(7)}(\xi)}{7!}}_{\in \frac{exp([0,1])}{7!}} \times \underbrace{(x - 1/2)^7}_{\leq (1/2)^{(-7)}}$$

$|\Delta_7(x, \xi)| \leq 1.46305781422e - 8$

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- Main idea:

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- Achieved so far:

$$\|\exp - p\|_\infty \leq \underbrace{\|\exp - T\|_\infty}_{\leq 1.46305781422e-8} + \underbrace{\|T - p\|_\infty}_{\text{bounding a polynomial}}$$

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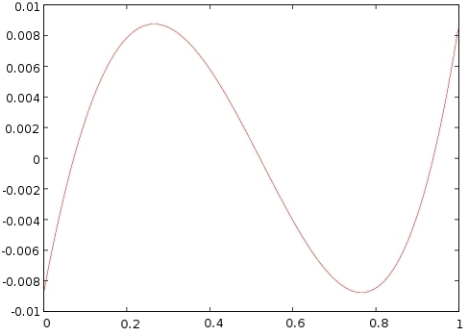

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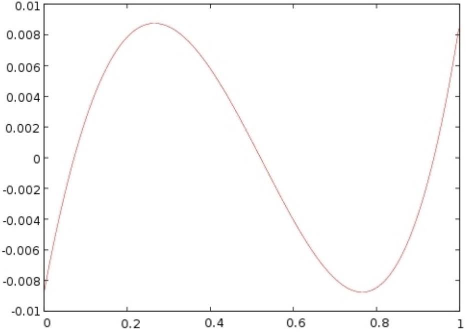

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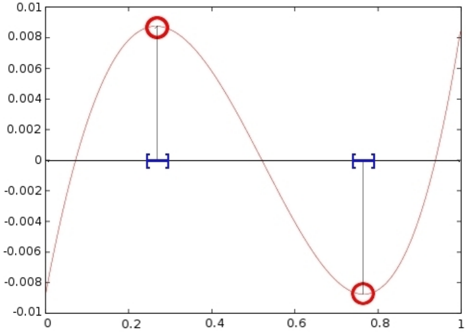

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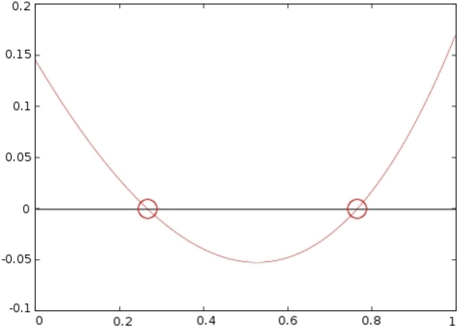

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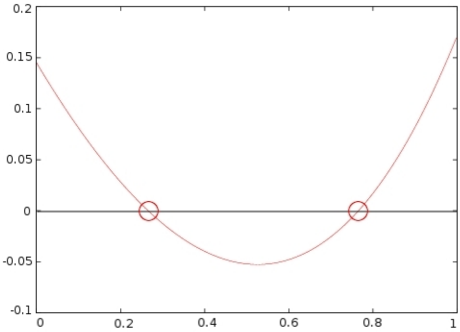

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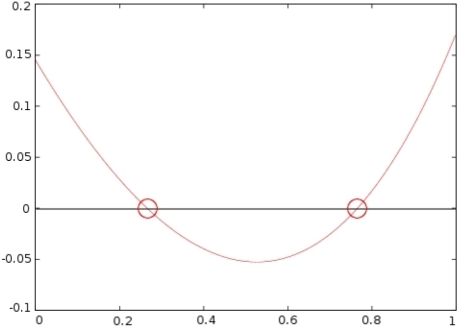

- Tightly bound the roots of the derivative
- Evaluate using interval arithmetic

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  - Sturm Theorem based strategies
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- Use a bisection strategy for isolating the roots
- Use dichotomy or Newton iteration process

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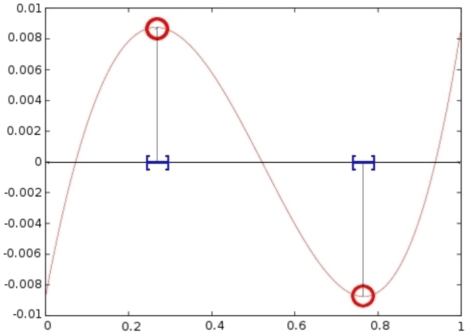

Our Approach - (3)Bounding the polynomial difference

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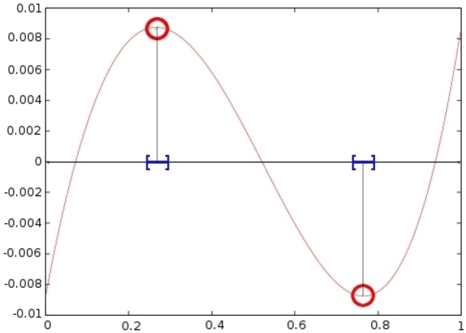

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  $r_1 \in [0.2657, 0.2659]$

  $r_2 \in [0.7652, 0.7654]$

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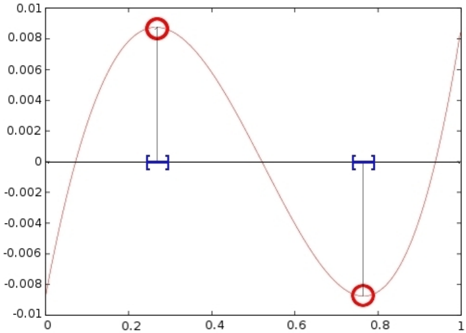

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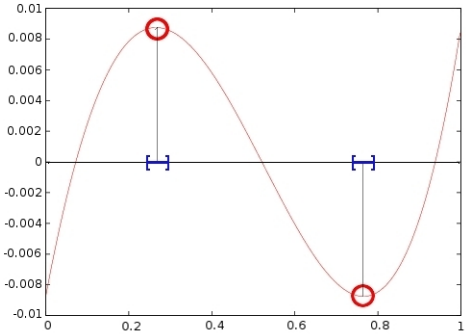

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- Evaluate using interval arithmetic

  $\|T - p\|_\infty \leq 0.0087566$

$\|f - p\|_\infty \simeq 0.00875606$

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Our example:

$$\|f - p\|_\infty \leq \underbrace{\|f - T\|_\infty}_{\leq 1.46305781422e-8} + \underbrace{\|T - p\|_\infty}_{\leq 0.0087566}$$

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By comparison, using interval arithmetic, we obtain $\|f-p\|_\infty \leq 0.2836$.

Results

Experiments were made on an Intel Pentium D 3.00GHz with a 2GB RAM.

f	$[a, b]$	d_p[1]	m[2]	acc[3]	time[4]
$\exp(x) - 1$	$[-0.25, 0.25]$	5	r	37.6	412
$\log_2(1 + x)$	$[-2^{-9}, 2^{-9}]$	7	r	83.3	2, 186
$\cos(x)$	$[-0.5, 0.25]$	15	r	19.5	2, 235
$\exp(x)$	$[-0.125, 0.125]$	25	r	42.3	7, 753
$\sin(x)$	$[-0.5, 0.5]$	9	a	21.5	520
$\exp(\cos(x)^2 + 1)$	$[1, 2]$	15	r	25.5	10, 984
$\tan(x)$	$[0.25, 0.5]$	10	r	26.0	1, 072
$x^{2.5}$	$[1, 2]$	7	r	15.5	1, 362

[1] Degree of p

[2] Error mode considered: a=absolute, r=relative

[3] Accuracy

[4] Timings in ms

Conclusion

- Safe and fast algorithm for bounding the supremum norm of the error functions
- Combination and reusal of various techniques (AD, polynomial roots isolation, interval arith)
- Absolute and Relative errors handled
- Faster and more accurate than other current approaches
- Future works:
  - Formal proof (AD, isolation of roots, multiple precision interval arithmetic are needed in the proof checker)
  - Replace "Taylor" polynomial with "Chebyshev-like interpolation polynomial"

Questions

Thank you for your attention!

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Experiments were made on an Intel Pentium D 3.00GHz with a 2GB RAM.

f	$[a, b]$	d_p[1]	m[2]	d_T[3]	acc[4]	time[5]
$\exp(x) - 1$	$[-0.25, 0.25]$	5	r	11	37.6	412
$\log_2(1 + x)$	$[-2^{-9}, 2^{-9}]$	7	r	23	83.3	2,186
$\cos(x)$	$[-0.5, 0.25]$	15	r	28	19.5	2,235
$\exp(x)$	$[-0.125, 0.125]$	25	r	41	42.3	7,753
$\sin(x)$	$[-0.5, 0.5]$	9	a	14	21.5	520
$\exp(\cos(x)^2 + 1)$	$[1, 2]$	15	r	60	25.5	10,984
$\tan(x)$	$[0.25, 0.5]$	10	r	21	26.0	1,072
$x^{2.5}$	$[1, 2]$	7	r	26	15.5	1,362

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