Stabilité des équilibres dans les modèles épidémiologiques

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1 Domaine de stabilité

Beaucoup de phénomènes naturels sont décrits par des équations différentielles et les plus simples d'entre elles sont les équations différentielles linéaires.

Le comportement des solutions d'une équation

$$X' = A\,X \tag{1}$$

est entièrement contrôlé par la localisation dans le plan complexe des racines du polynôme caractéristique P de A

$$X^p + a_{p-1}\,X^{p-1} + \cdots + a_0 = 0. \tag{2}$$

En effet si $x_1, ..., x_r$ sont les racines complexes de P de multiplicités $\mu_1 ..., \mu_r$, les fonctions

$$t^j e^{x_i t}, i = 1, ...r, j = 0, ..., \mu_i - 1$$

forment une base des solutions de l'Equation (1). Quand tous les x_i ont une partie réelle négative, toutes les solutions de l'Equation (1) tendent vers 0 quand t tend vers $+\infty$. C'est la raison pour laquelle l'ensemble des polynômes de degré p ayant toutes leurs racines avec partie réelle négative est appelé le **domaine de stabilité** de degré p.

1.1 En général

Le domaine de stabilité a une description très explicite.

Théorème 1. [Liénard/Chipart] *Le polynôme*

$$P = a_p X^p + \cdots + a_0, a_p > 0,$$

appartient au domaine de stabilité de degré p si et seulement si $a_i > 0$, $i = 0, ..., p$, et

$$\begin{cases} \mathrm{sRes}_{m-1}(F, G) > 0, ..., \mathrm{sRes}_0(F, G) > 0 & \text{si } p = 2m, \\ \mathrm{sRes}_m(XG, F) > 0, ..., \mathrm{sRes}_0(XG, F) > 0 & \text{si } p = 2m+1 \end{cases}$$

avec $P(X) = F(X^2) + XG(X^2)$.

Les sous-résultants sont des déterminants extraits de la matrice de Sylvester matrix, dont les coefficients sont des coefficients de F et G (donc de P).

Exemples de domaine de stabilité

a) degré 1 et degré 2: tous les coefficients de P sont positifs

b) degré 3:

$P = a\,X^3 + b\,X^2 + c\,X + d,$

$F = b\,X + d,\ G = a\,X + c,$

le domaine de stabilité est caractérisé par

$$a > 0, b > 0, c > 0, d > 0, R = d\,(b\,c\,d - a\,d) > 0,$$

où

$$R = d\,(b\,c - a\,d) = \begin{vmatrix} a & c & 0 \\ 0 & b & d \\ b & d & 0 \end{vmatrix}$$

c) degré 4:

$P = a X^4 + b X^3 + c X^2 + d X + e,$

$F = a X^2 + c X + e,\ G = b X + d,$

le domaine de stabilité est caractérisé par

$$a > 0, b > 0, c > 0, d > 0, e > 0, R > 0$$

$$R = -b^2 e - a d^2 + b c\ d = \begin{vmatrix} a & c & e \\ 0 & b & d \\ b & d & 0 \end{vmatrix}$$

d) si le polynôme factorise, il est plus simple de regarder les conditions de stabilité sur chaque facteur.

1.2 Un cas particulier utile

E-matrice : tous les coefficients diagonaux sont < 0 et tous les éléments non-diagonaux sont $\geqslant 0$.

Soit A une E-matrice , le polynôme caractéristique de A appartient au domaine de stabilité si et seulement si $(-1)^i A_i > 0$ où A_i est le mineur principal d'ordre i de A.

• 2 Phénomènes non-linéaires en épidémiologie

Les phénomènes épidémiologiques ne sont pas controlés par des équations différentielles linéaires mais bilinéaires : la maladie est transmise proportionnellement aux nombre des infectés I et au nombre des susceptibles S, et donc des termes $I\,S$ vont apparaître dans les équations.

On peut faire les observations suivantes sur des modèles épidémiologiques très simples

- il y a des équilibres sans maladie et des équilibres endémiques (un certain pourcentage de malades est constamment présent)

- ces équilibres sont ou non stables. Ici stable signifie intuitvement « persistant au cours du temps même en présence de petites perturbations".

Quand on considère des familles de modèlels à paramètres, l' existence d'équilibres sans maladie ou endémiques et leur nature stable ou instable tombe dans le domaine des techniques semi-algébriques. Il faut identifier algébriquement les équilibres (typiquement par un calcul de base de Groebner), puis étudier leur stabilité: les conditions sur les paramètres qui caractérisent l'existence d'un équilibre sans maladie ou d'un équilibre endémique, celles pour lesquelles le polynôme caractéristique de la matrice jacobienne des dérivées partielles (pour à un équilibre donné) est dans le domaine de stabilité. Il sera particulièrement intéressant d'étudier la tradition entre une situation et une autre (apparition d'un équilibre endémique, par exemple).

• 3 SIR model

Three groups are identified in the population and their proportion (%) in the total population is considered

→ S: % of susceptible

→ I: % of infectious

→ R: % of immunes (recovered from disease)

The parameters of the model

→ β: parameter of transmission

→ μ: rate of death =rate of birth

→ γ: rate of loss of immunity

The scheme of transmission

$$\xrightarrow{\mu} \quad \underset{\downarrow \mu S}{S} \quad \xrightarrow{\beta IS} \quad \underset{\downarrow \mu I}{I} \quad \xrightarrow{\gamma I} \quad \underset{\downarrow \mu R}{R}$$

The change in the proportion of susceptible comes from

- adding new born
- removing dead susceptible
- removing susceptible becoming infected

The differential system $\dot{X} = f(X)$ is

$$\begin{aligned} \dot{I} &= \beta\, I\, S - (\mu + \gamma)\, I \\ \dot{R} &= \gamma\, I - \mu\, R \\ \dot{S} &= \mu - \mu\, S - \beta\, I\, S \end{aligned}$$

and consists of three equations f_1, f_2, f_3 in three unknowns (bilinear terms in red).

A few properties of this system

(H_1) Positive invariance of the cone $\mathbb{R}^n_+$:

Suppose all the parameters are positive and all the variables are non negative then $I = 0$ implies $\dot{I} \geqslant 0$ (same thing for R and S)

(H_2) Invariance of the disease free space

$I = R = 0$ implies $\dot{I} = \dot{R} = 0$

Equilibria correspond to

$$\begin{aligned} 0 &= \beta I S - (\mu + \gamma) I, \\ 0 &= \gamma I - \mu R, \\ 0 &= \mu - \mu S - \beta I S. \end{aligned}$$

Epidemiological equilibria are equilibria with all their coordinates between 0 and 1.

There is an immediate solution $X_1 = (0, 0, 1)$: $I_1 = R_1 = 0, S_1 = 1,$ which is **disease free**.

In order to study its stability, consider the characteristic polynomial of the jacobian matrix $\partial_X f$ obtained by taking partial derivatives with respect to I, R, S

$$\begin{bmatrix} \beta - \gamma - \mu & 0 & 0 \\ \gamma & -\mu & 0 \\ -\beta & 0 & -\mu \end{bmatrix}.$$

The block structure of the matrix follows from (H_2).

Its characteristic polynomial of degree 3 factorizes in three factors of degree one

$$(x + \mu)^2 (x + \gamma + \mu - \beta).$$

So the diseasefree equilibrium is stable if

$$\beta < \beta_0 = \gamma + \mu$$

that is if the rate of transmission β is small enough with respect to the other parameters γ and μ. In this case $\partial_X f$ is an E matrix !

What happens if β goes through β_0?

• 4 Transcritical bifurcation

We consider a one parameter system of differential equations $\dot{X} = f(u, X)$, where X is a finite set of k variables, u is a variable, and $f = f_1, ..., f_k$ is a finite number of polynomials in u, X.

Définition 2. $(0, X_0)$ is a **transcritical bifurcation** of $\dot{X} = f(u, X)$ if there exists a neigbourhood V of X_0 such that

- X_0 only equilibrium of $\dot{X} = f(0, X)$ in V
- 0 unique eigenvalue of the Jacobian matrix $\partial_X f(0, X_0)$ with 0 real part, and simple
- exist $\varepsilon > 0$, smooth maps $\psi_1, \psi_2 : (-\varepsilon, \varepsilon) \to \mathbb{R}^k$ and $\lambda_1, \lambda_2 : (-\varepsilon, \varepsilon) \to \mathbb{R}$ with
  - $\psi_1(0) = \psi_2(0) = X_0$, $\lambda_1(0) = \lambda_2(0) = 0$,
  - for any $u \in (-\varepsilon, \varepsilon)$, $\psi_1(u), \psi_2(u)$ only equilibria of $\dot{X} = f(u, X)$ in V
  - $\lambda_1(u)\ \lambda_2(u) < 0$ for $u \neq 0$, and, $\lambda_i(u)$ eigenvalue of $\partial_X f(u, \psi_i(u))$, $i = 1, 2$. □

It is rather easy to prove that **transcritical bifurcation** correspond to **exchange of stability** at 0 between equilibrium $X_1 = \psi_1(u)$ and $X_2 = \psi_2(u)$ in the special case when 0 is the only eigenvalue of $\partial_X f(0, X_0)$ with non-negative real part: when $u < 0$ equilibrium X_1 is stable and equilibriumX_2 is unstable, while when $u > 0$ equilibrium X_1 is unstable and equilibriumX_2 is stable (if $\lambda_1(u) > 0$ for $u < 0$).

The following theorem, which is a special case of a theorem by Sotomayor [8] gives sufficient conditions for transcritical bifurcations.

Théorème 3. *Let* $\dot{X} = f(u, X)$, and $0, X_0$ *be such that* X_0 *is an equilibrium of* $\dot{X} = f(0, X)$. *Assume that* 0 *is a simple eigenvalue of* $\partial_X f(0, X_0)$ *and let* V_r *and* V_ℓ *be eigenvectors associated to the eigenvalue* 0 *at the left and the right of* $\partial_X f(0, X_0)$. *If*

$$\begin{aligned} V_\ell \cdot \partial_u f(0, X_0) &= 0, \\ V_\ell \cdot \partial_u \partial_X f(0, X_0) \cdot V_r &\neq 0, \\ V_\ell \cdot \partial_X^2 f(0, X_0)\,(V_r, V_r) &\neq 0, \end{aligned}$$

then $(0, X_0)$ *is a transcritical bifurcation of* $\dot{X} = f(u, X)$.

Typical case

Take $k=1$,

$$\dot{X} = a\,u\,X + b\,X^2 + \text{higher order terms in } u \text{ and } X$$

$(0,0)$ satifsies the hypothesis of Sotomayor theorem since, noting that $V_\ell = V_r = 1$,

$$\begin{aligned} \partial_u f(0,0) &= 0, \\ \partial_u \partial_X f(0,0) = a &\neq 0, \\ \partial_X^2 f(0,0) = 2\,b &\neq 0. \end{aligned}$$

It is easy to check that it is a transcritical bifurcation since the are two equilibria which are

$$X_1 = 0 + ..., \; X_2 = -\frac{u\,a}{b} + \cdots.$$

The derivative is $a\,u + 2\,b\,X + \cdots$ and its value at X_1 is $a\,u + \cdots$ while its value at X_2 is $-a\,u + \cdots$.

- **Proof of Theorem**

  Reduce the general case to the typical case.

  Suppose that X_0 is an equilibrium and 0 is an eigenvalue of $\partial_X f$. Let V_r and V_ℓ be eigenvectors associated to the eigenvalue 0 at the left and the right of $\partial_X f(0,\, X_0)$. Define

$$\begin{aligned} T &= V_\ell \cdot X \\ g(u,X) &= V_\ell \cdot f(u,X) \\ \dot{T} &= g(u, X_0 + T V_r) = h(u,T). \end{aligned}$$

  The system $\dot{X} = f(u,X)$ satisfies conditions

$$\begin{aligned} V_\ell \cdot \partial_u f(0,X_0) &= 0, \\ V_\ell \cdot \partial_u \partial_X f(0,X_0) \cdot V_r = a &\neq 0, \\ V_\ell \cdot \partial_X^2 f(0,X_0)\,(V_r,V_r) = 2\,b &\neq 0, \end{aligned}$$

  if and only if

$$h(u,T) = a\,u\,T + b\,T^2 + \text{higher order terms},$$

  i.e. if h is of the typical case.

• 5 SIR model and transcritical bifurcation

Let us apply the method of transcritical bifurcation for the variable $u = \beta - \beta_0$ at 0 for the model SIR. In this case there is only one equilibrium $X_0 = (1, 0, 0)$, the jacobian matrix $\partial_X f$

$$\begin{bmatrix} 0 & 0 & 0 \\ \gamma & -\mu & 0 \\ -\beta_0 & 0 & -\mu \end{bmatrix}$$

has characteristic polynomial

$$(x+\mu)^2\, x.$$

The only non zero eigenvalue is $-\mu$ which has negative real part.

The eigenvalue 0 has associated eigenvectors to the right and to the left

$$V_\ell = [1 \;\; 0 \;\; 0], V_r = \begin{bmatrix} 1 \\ \frac{\gamma}{\mu} \\ -\frac{\beta_0}{\mu} \end{bmatrix}.$$

We compute

$$\begin{aligned}
\partial_u f_1(0, X_0) &= 0, \\
\partial_X \partial_u f_1(0, X_0) &= [1 \;\; 0 \;\; 0], \\
\partial_X^2 f_1(0, X_0) &= \begin{bmatrix} 0 & 0 & \beta_0 \\ 0 & 0 & 0 \\ \beta_0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

Hence

$$\begin{aligned}
V_\ell \cdot \partial_u f(0, X_0) = \partial_u f_1(0, X_0) &= 0, \\
V_\ell \cdot \partial_u \partial_X f(0, X_0) \cdot V_r &= \\
\partial_u \partial_X f_1(0, X_0) \cdot V_r &= 1 \neq 0, \\
V_\ell \cdot \partial_X^2 f(0, X_0)\,(V_r, V_r) &= \\
\partial_X^2 f_1(0, X_0)\,(V_r, V_r) &= 2\,\beta_0 \left(\frac{-\beta_0}{\mu} \right) \neq 0.
\end{aligned}$$

So the hypothesis of the theorem of Sotomayor are satisfied and the bifurcation if transcritical.

The second equilibrium is given by the proof of Sotomayor theorem

$$\begin{aligned} T &= V_\ell \cdot X = I \\ g(u,X) &= V_\ell \cdot f(u,X) = \beta\, I\, S - \beta_0 I \\ \dot{I} &= \beta\, I \left(1 - I\frac{\beta_0}{\mu}\right) - \beta_0\, I \\ &= (\beta - \beta_0)\, I - \frac{\beta_0\, \beta}{\mu}\, I^2: \end{aligned}$$

If $\dot{I} = 0$, the I coordinate of the second equilibrium is

$$\frac{\mu(\beta - \beta_0)}{\beta_0\, \beta}$$

and it follows that the second equilibrium X_2 is equal to

$$S_2 = \frac{\beta_0}{\beta},\ I_2 = \frac{\mu\,(\beta - \beta_0)}{\beta_0\ \beta},\ R_2 = \frac{\gamma\,(\beta - \beta_0)}{\beta_0\ \beta}.$$

Note that X_2 is an epidemiological equilibrium only when $\beta > \beta_0$ (when $\beta < \beta_0$, $S_2 > 1$: remember that S_2 is a %).

The conclusion is that the situation is entirely controlled by the value of β

- if $\beta < \beta_0 = \gamma + \mu$, there is only one epidemiological equilibrium, which is disease free and stable (the second equilibrium exists and is unstable but is not epidemiological since $S > 1$)
- if $\beta = \beta_0 = \gamma + \mu$, there is only one epidemiological equilibrium, which is disease free and corresponds to a transcritical bifurcation
- if $\beta > \beta_0 = \gamma + \mu$, there are two epidemiological equilibria one disease free and unstable, the other endemic and stable.

• 6 The SEIT model

(Driessche & Watmough 2002)

Four groups are identified in the population and their proportion (%) in the total population is considered

→ E: % of exposed

→ I: % of infectious

→ S: % of susceptible

→ T: % of under treatment

The parameters of the model

→ β_1: parameter of transmission for the susceptible

→ β_2: parameter of transmission for those undertreatment $\beta_1 > \beta_2$

→ μ: rate of death =rate of birth

→ ν: rate from exposed to infectious

→ r_1: rate of treatment for the exposed

→ r_2: rate of treatment for the infectious

→ q: : rate of infectious successfully treated

The scheme of transmission

The differential system $\dot{X} = f(X)$ is

$$\begin{aligned}
\dot{E} &= \beta_1 I S + \beta_2 I T - (\mu + \nu + r_1) E + (1-q) r_2 I \\
\dot{I} &= \nu E - (\mu + r_2) I \\
\dot{S} &= \mu - \mu S - \beta_1 I S \\
\dot{T} &= r_1 E + q r_2 I - \beta_2 I T - \mu T
\end{aligned}$$

and consists of three equations f_1, f_2, f_3, f_4 in four unknowns (bilinear terms in red).

Note that, once again,

(H_1) (positive invariance) is true

(H_2) (invariance of the disease free space $E = I = 0$) is true.

Equilibria correspond to

$$\begin{aligned}
0 &= \beta_1 I S + \beta_2 I T - (\mu + \nu + r_1) E + (1-q) r_2 I \\
0 &= \nu E - (\mu + r_2) I \\
0 &= \mu - \mu S - \beta_1 I S \\
0 &= r_1 E + q r_2 I - \beta_2 I T - \mu T
\end{aligned}$$

Epidemiological equilibria are equilibria with all their coordinates between 0 and 1.

There is an immediate solution $X_1 = (0, 0, 1, 0)$: $E_1 = I_1 = 0$, $S_1 = 1$, $T_1 = 0$ which is **disease free**.

In order to study its stability, consider the characteristic polynomial of the jacobian matrix $\partial_X f$ obtained by taking partial derivatives with respect to E, I, S, T

$$\begin{bmatrix} -\mu-\nu-r_1 & \beta_1+(1-q)\,r_2 & 0 & 0 \\ \nu & -\mu-r_2 & 0 & 0 \\ 0 & \beta_1 & -\mu & 0 \\ r_1 & q\,r_2 & 0 & -\mu \end{bmatrix}.$$

The block structure of the matrix follows from (H_2).

Its characteristic polynomial of degree 4 factorizes in two factors of degree one and one factor of degree 2

$$(x+\mu)^2\,(x^2+(r_1+2\,\mu+\nu+r_2)x+((\mu+r_2)\,(\mu+\nu+r_1)-\nu\,(\beta_1+(1-q)\,r_2)\,)$$

So the diseasefree equilibrium is stable if

$$\beta_1 < \beta_0 = \frac{\mu\,(r_1+r_2+\nu\,+\mu)+\nu q\,r_2+r_1\,r_2}{\nu}$$

that is if the rate of transmission β_1 is small enough with respect to the other parameters.

Let us apply the method of transcritical bifurcation for the variable $u = \beta_1 - \beta_0$ at 0. In this case there is only one equilibrium $X_0 = (0, 0, 1, 0)$, the jacobian matrix

$$\begin{bmatrix} -\mu-\nu-r_1 & \beta_0+(1-q)\,r_2 & 0 & 0 \\ \nu & -\mu-r_2 & 0 & 0 \\ r_1 & q\,r_2 & -\mu & 0 \\ 0 & \beta_0 & 0 & -\mu \end{bmatrix}.$$

has characteristic polynomial

$$(x+\mu)^2\,(x+r_1+2\,\mu+\nu+r_2)\,x.$$

The only non zero eigenvalues are $-\mu$ and $-(r_1+2\,\mu+\nu+r_2)$ which have negative real part.

The eigenvalue 0 has associated eigenvectors to the right and to the left

$$V_\ell = \begin{bmatrix} 1 & \dfrac{\mu+\nu+r_1}{\nu} & 0 & 0 \end{bmatrix}, V_r = \begin{bmatrix} 1 \\ \dfrac{\beta_0}{\mu} \\ \dfrac{\mu+r_2}{\nu} \\ \dfrac{r_1\,(\mu+r_2)+\nu\,q\,r_2}{\mu\,\nu} \end{bmatrix}.$$

We compute

$$\begin{aligned} g &= f_1 + \frac{\mu+\nu+r_1}{\nu} f_2 \\ &= \beta_1 I S + \beta_2 I T - \beta_0 I \\ \partial_u g(0, X_0) &= 0, \\ \partial_X \partial_u g(0, X_0) &= [0 \;\; 0 \;\; 1 \;\; 0], \\ \partial_X^2 g(0, X_0) &= \begin{bmatrix} 0 & 0 & \beta_0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta_0 & 0 & 0 & \beta_2 \\ 0 & 0 & \beta_2 & 0 \end{bmatrix}. \end{aligned}$$

Hence

$$\begin{aligned} V_\ell \cdot \partial_u f(0, X_0) = \partial_u g(0, X_0) &= 0, \\ V_\ell \cdot \partial_u \partial_X f(0, X_0) \cdot V_r &= \\ \partial_u \partial_X g(0, X_0) \cdot V_r &= 1 \neq 0, \\ V_\ell \cdot \partial_X^2 f(0, X_0)(V_r, V_r) &= \\ \partial_X^2 g(0, X_0)(V_r, V_r) &= 2\left(\frac{\beta_0^2}{\mu} + \frac{\beta_2 (r_1(\mu + r_2) + \nu q r_2)}{\mu \nu} \right) \neq 0. \end{aligned}$$

So the hypothesis of the theorem of Sotomayor are satisfied and the bifurcation if transcritical.

Using the proof of Sotomayor theorem, we define

$$\begin{aligned} V &= V_\ell \cdot X = E + \frac{\mu+\nu+r_1}{\nu} I \\ g(X) &= V_\ell \cdot f(u,X) = \beta_1 I S + \beta_2 I T - \beta_0 I \end{aligned}$$

and it follows by an easy elimination that S_2 is a root of the second degree equation $A(S)$:

$$\beta_2 (S-1)(r_2 \nu q + r_1 \mu + r_1 r_2) + \nu (\beta_1 S + \beta_2 (S-1))(\beta_1 S - \beta_0)$$

and the other variables are rational expression of S_2

$$\begin{aligned} E_2 &= \frac{(\mu + r_2)\,\mu(\beta_1 S_2 - \beta_0)}{\beta_2(\nu(\beta_1 S_2 - \beta_0) + r_1\ \mu + r_2 \nu q + r_1\ r_2)} \\ I_2 &= \frac{\mu\ \nu(\beta_1 S_2 - \beta_0)}{\beta_2(\nu(\beta_1 S_2 - \beta_0) + r_1\ \mu + r_2 \nu q + r_1\ r_2)} \\ T_2 &= \frac{-(\beta_1 S_2 - \beta_0)}{\beta_2}. \end{aligned}$$

Note that the endemic equlibrium is not expressed rationnally in terms of the parameters !

When $\beta_1 > \beta_0$, it is easy to see that there is a value S_2 between 0 and β_0/β_1 by evaluating A at 0 and β_0/β_1, so that, for this value of S_2, E_2, I_2 and T_2 are positive.

When $\beta_1 < \beta_0$, it is not complicated to see that there is no value S_2 between 0 and 1 by evaluating A at 0 and 1, so that there is no endemic equilibrium.

The conclusion is that the situation is entirely controlled by the value of β_1

- if $\beta_1 < \beta_0$, there is only one epidemiological equilibrium, which is disease free and stable (the second equilibrium exists and is unstable but is not epidemiological since $S_2 > 1$)
- if $\beta_1 = \beta_0$, there is only one epidemiological equilibrium, which is disease free and corresponds to a transcritical bifurcation
- if $\beta_1 > \beta_0$, there are two epidemiological equilibria one disease free and unstable, the other endemic and stable.

7 More models

The very simple scheme we have seen in this model appears again and again in more sophisticated epidemiological models:

- SEIR where the period of infection is taken into account
- SEIT where some people are under treatment
- SEIRS where the immunity can be lost after a while
- MSEIRS where new born are protected by their mother immunization
- MSEITV when part of the population is vaccinated.

The more complicated the model is, the more fruitful it is to use computer algebra (example SIR can be easily computed by hand). Model MSEITV is new.

All of these models have the same behaviour: there is a value of β_0 given as a function of the other parameters such that

- if $\beta < \beta_0$, there is only one epidemiological equilibrium, which is disease free and stable (the second equilibrium exists and is unstable but is not epidemiological since $S > 1$)
- if $\beta = \beta_0$, there is only one epidemiological equilibrium, which is disease free and corresponds to a transcritical bifurcation
- if $\beta > \beta_0$, there are two epidemiological equilibria one disease free and unstable, the other endemic and stable.

Otto and M'hamed started to study a more sophisticated model proposed by Thierry, involving resistance to antibiotics. In this case there are much more than two equilibria, but transcritical bifurcations appear in all transitions.

• 8 A few comments

– In chemistry and ecology and many other domains there are bilinear systems of differential equations depending on parameters, and people study equilibria and their stability. See for example papers by Pierre Auger in ecology or recent work of Igor Klep on chemistry.

In epidemiology and other parts of biology, transcritical equilibria are typical.

In chemistry a typical situation is to have one single stable equilibrium. The system looks very similar to the ones we have in epidemiology but some crucial signs in the equations are not the same.

In ecology there are significant differences in qualitative behaviour between "predator prey systems" (the predator gets bigger when he eats the prey) and "competition" (two species competing for the same resources, they can both get slim).

So, signs do matter ...

– In epidemiology the modelization phase guiding the public health policies (example: strategies of vaccination) is better and better acknowledged, see lecture of W. Orenstein from Emory Vaccine Center (US). [7].

– Africa needs specific techniques adapted to its epidemiological situation. For example, with a number of child per woman over 7, and a population multiplied by 5 in one century, the population of Niger cannot be considered as constant. Also the migration in town of undervaccinated populations at some periods of the year creates specific problems [6]. It is important to study them. And it is important to have local experts.

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