Divide and Conquer Roadmap Algorithms for Real Algebraic Sets

Complexity issues

M. Safey El Din

INRIA Paris-Rocquencourt SALSA Project-team
Université Pierre et Marie Curie

Joint work with E. Schost
University of Western Ontario, Canada

What is it?

$f_1, \ldots, f_s$ in $\mathbb{Q}[X_1, \ldots, X_n]$ of degree bounded by D.
$V = V(f_1, \ldots, f_s) \subset \mathbb{C}^n$.

A roadmap $\mathcal{R}$ is an algebraic curve included in $\mathcal{V}$ which has a connected and non-empty intersection with each connected component of $\mathcal{V} \cap \mathbb{R}^n$.

Introduced initially for solving Robot Motion Planning Problems

- Historical problem: On the piano mover's problem (Schwartz/Sharir, 1983)
- See the book: *Planning algorithms* (S.M. LaValle, Univ. of Illinois, Camb. Univ. Press, `http://planning.cs.uiuc.edu/`)

Core idea: Reduce the connectivity decision problems in general semi-algebraic sets to connectivity decision problems in semi-algebraic sets of dimension ≤ 1.

Motivations

- Real solving polynomial systems of inequations: Size of the output will be soon a problem (implementations based on the critical point method). Can be important for solving classification problems via the discriminant variety (Lazard/Rouillier)
- Computational geometry:
  - Work of Everett H., Lazard D., Lazard S., S. on Voronoi diagrams of 3 generic lines in $\mathbb{R}^3$: the topology of the Voronoi diagram is fully described by ad hoc methods
  - Counting the number of connected components of semi-algebraic sets can be also usefull (enumerative geometry problems studied by Lazard, Goaoc and Petitjean).
- Algorithmic semi-algebraic geometry: Roadmaps are tools for
  - Deciding of two given points lie in the same connected components of a semi-algebraic set
  - Counting the number of connected components of semi-algebraic sets
  - Obtaining a semi-algebraic description of these connected components

State of the Art

- **Canny's strategy (1987)**
  Degree of the output: $D^{\mathcal{O}(n^2)}$ (worst case reached $D^{\frac{n(n-1)}{2}}$)
  Complexity $D^{\mathcal{O}(n^4)}$ (deterministic algorithm)
  Monte-carlo version of this algorithm has a complexity $D^{\mathcal{O}(n^2)}$
  Many other works [Grigoriev/Vorobjov, Heintz/Roy/Solerno, Gournay Risler and Basu/Pollack/Roy $\rightarrow D^{\mathcal{O}(n^2)}$]
  Practical considerations introduced by Mezzarrobba/S. in smooth situations (2006) $D^{\mathcal{O}(nd)}$ (where d is the dimension of the studied variety)
- **Algebro-Differential approach:** intensively developped by H. Hong and R. Quinn (based on numerical integration of Morse/Smale vector fields).
  See also D'Acunto's research projects (and his works with Kurdyka) based on the study of metric properties of semi-algebraic sets.
  See also Yomdin and Comte's book: Tame Geometry.

Summary of our results

Let $V \subset \mathbb{C}^n$ be an algebraic variety defined by $f_1 = \cdots = f_s = 0$ and $D = \max(\deg(f_i), 1 \leq i \leq s)$.

First result: There exists a deterministic algorithm computing a roadmap of $V \cap \mathbb{R}^n$ of degree bounded by $D^{\mathcal{O}(n\sqrt{n})}$

Remember that Canny's strategy outputs a roadmap of degree bounded $D^{\mathcal{O}(n^2)}$

This algorithm is based on a new connectivity result generalizing the one of Canny

We are not able to obtain satisfactory bounds on the arithmetic complexity of this algorithm.

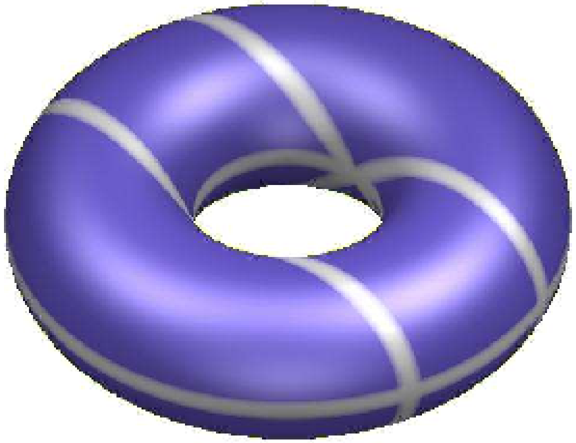
Second result: There exists a *probabilistic* algorithm computing a roadmap of $V \cap \mathbb{R}^n$ whose arithmetic complexity is polynomial in $D^{n\sqrt{n}}$.

This improves the original result of Canny ($D^{\mathcal{O}(n^2)}$)

Canny's strategy in a nutshell – Canny routine – $D^{\mathcal{O}(nd)}$

$\langle f_1, \ldots, f_s \rangle$ radical defining $V \subset \mathbb{C}^n$, $\sharp\mathrm{Sing}(V) < \infty$, $\overline{V \setminus \mathrm{Sing}(V)}$ equi-dimensional of dimension d, and $V \cap \mathbb{R}^n$ is compact.

- Compute the critical locus $\mathcal{C}$ of the projection onto (X_1, X_2) restricted to $\mathcal{V}$
  $\longrightarrow \deg(\mathcal{C}) \leq D^{n-1}$
- Compute the critical values of the projection on X_1 restricted to $\mathcal{C}$
  These are encoded as the roots of a polynomial $P \in \mathbb{Q}[X_1]$
- Recursive call to the algorithm by instantiating X_1 to each of these critical values
  Compute modulo P which has degree at most $D^{\mathcal{O}(n)}$ (with $D = \max(\deg(f_i))$)



Basu/Pollack/Roy: Algebraic manipulations for reducing the computation of a roadmap in a general real algebraic set to computing a roadmap in a smooth hypersurface whose real counterpart is compact

Connectivity result

Let $V \subset \mathbb{C}^n$ be an algebraic variety and a finite set of points $\mathcal{P}_0 \subset V \cap \mathbb{R}^n$.

Consider the projections $\pi_i : (x_1, \ldots, x_n) \in \mathbb{C}^n \rightarrow (x_1, \ldots, x_i)$

Denote by $\mathcal{S}(\pi_i, V)$ the union of

- the set of critical points of the restriction of π_i to V
- the set of singular points of V.

We suppose that

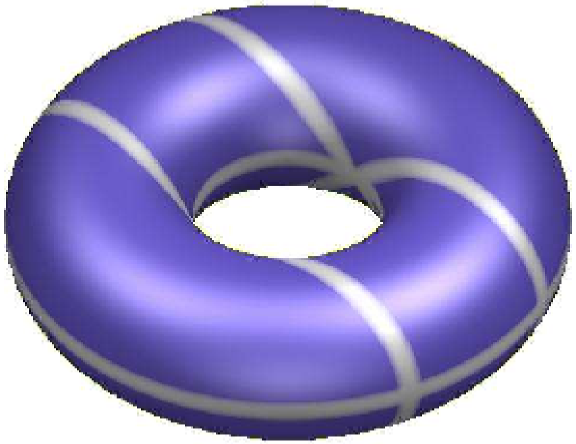
- $\mathrm{Sing}(V)$ is finite, $\overline{V \setminus \mathrm{Sing}(V)}$ is equi-dimensional and $V \cap \mathbb{R}^n$ is compact
- $\mathcal{S}(\pi_1, \mathcal{S}(\pi_i, V)) \cup \mathcal{S}(\pi_1, V)$ is finite. This set of points is denoted by $\mathcal{P}_i$

In the sequel $\mathcal{F}_i$ denotes $\pi_{i-1}^{-1}(\pi_{i-1}(\mathcal{P}_i \cup \mathcal{P}_0)) \cap V$.

Then, each connected component of $V \cap \mathbb{R}^n$ has a non-empty and connected intersection with $\mathcal{S}(\pi_i, V) \cup \mathcal{F}_{i-1}$.

Following Basu/Pollack/Roy, this theorem can be extended to cases where V is defined by polynomials with coefficients in $\mathbb{Q}(\varepsilon_1, \ldots, \varepsilon_k)$ (where ε_i are infinitesimals)

Examples and additional statement



Let $\mathcal{R}_1 \cup \mathcal{R}_2$ be a roadmap of dimension i of V. Let $\mathcal{R}'_1$ (resp. $\mathcal{R}'_2$) be a roadmap of $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) of dimension $j_1 < i$ (resp. $j_2 < i$). If $\mathcal{R}_1 \cap \mathcal{R}_2 \subset \mathcal{R}'_1$ and $\mathcal{R}_1 \cap \mathcal{R}_2 \subset \mathcal{R}'_2$, $\mathcal{R}'_1 \cup \mathcal{R}'_2$ is a roadmap of V of dimension $\max(j_1, j_2)$.

Ensuring the assumptions of the connectivity result

Let $\{f = 0\} \subset \mathbb{C}^n$ whose set of singular points has dimension at most 0. There exists a Zariski-closed subset $\mathcal{A} \subsetneq GL_n(\mathbb{C})$ such that for all $\mathbf{A} \in GL_n(\mathbb{Q}) \setminus \mathcal{A}$:

- $\mathcal{S}(\pi_i, \{f^{\mathbf{A}} = 0\}) \setminus \mathrm{Sing}(\mathcal{S}(\pi_i, \{f^{\mathbf{A}} = 0\}))$ is equi-dimensional, of dimension $i - 1$ and smooth [Bank/Giusti/Heintz/M'Bakop].
- For all $x \in \mathbb{C}^{i-1}$, $\pi_{i-1}^{-1}(x) \cap \{f^{\mathbf{A}} = 0\}$ has dimension $n - i$, it has at most a finite set of singular points.
- For all $x \in \mathbb{C}^{i-1}$, $\pi_{i-1}^{-1}(x) \cap \mathcal{S}(\pi_i, \{f^{\mathbf{A}} = 0\})$ has dimension at most 0 [S./Schost 03]
- $\mathcal{S}(\pi_1, \mathcal{S}(\pi_i, \{f^{\mathbf{A}} = 0\}))$ has dimension at most 0.
  Here, using transversality results (such as Sard's theorem or Thom's transversality theorems) are not sufficient since π_1 is not here a *generic* projection relatively to π_i and $\mathcal{S}(\pi_i, \{f^{\mathbf{A}} = 0\})$!

The algorithm – Roadmap routine

Input: a set of polynomials $f_1, \ldots, f_s$ in $\mathbb{Q}[X_1, \ldots, X_n]$ and a finite set of points $\mathcal{P}_0 \subset V(f_1, \ldots, f_s)$
Output: a roadmap of $V(f_1, \ldots, f_s) \cap \mathbb{R}^n$.

- Reduction to the case of a smooth hypersurface $\mathcal{H}$ with a bounded real counterpart defined by $f = 0$ (following Basu/Pollack/Roy)
- Set $i = \lfloor \sqrt{n} \rfloor$
- Compute $\mathcal{P} = \mathcal{S}(\pi_1, \mathcal{S}(\pi_i, \mathcal{H})) \cup \mathcal{S}(\pi_1, \mathcal{H})$
- Compute $\mathcal{F} = \pi_{i-1}(\mathcal{P} \cup \mathcal{P}_0)$
  Remember that $\mathcal{S}(\pi_i, \mathcal{H}) \cup \left(\pi_{i-1}^{-1}(\mathcal{F}) \cap \mathcal{H}\right)$ has a non-empty and connected intersection with each connected component of the real counterpart of $\mathcal{H}$.
  Call Canny to compute a roadmap in $\mathcal{S}(\pi_i, \mathcal{H})$ and recursive call to Roadmap with input $\pi_{i-1}^{-1}(\mathcal{F}) \cap \mathcal{H}$
- Return Canny$([f^{\mathbf{A}}, \frac{\partial f^{\mathbf{A}}}{\partial X_n}, \ldots, \frac{\partial f^{\mathbf{A}}}{\partial X_{i+1}}], (\mathcal{F} \cap \mathcal{S}(\pi_i, \mathcal{H}))^{\mathbf{A}} \cup \mathcal{P}_0^{\mathbf{A}})^{\mathbf{A}^{-1}} \bigcup \cup_{p \in \mathcal{F}}$ Roadmap$([\phi_p(f)])$

Complexity estimates

DegreeCanny(Degree, n, dim) and DegreeRoadmap(Degree, n, dim)

DegreeCanny(D, n, $\lfloor\sqrt{n}\rfloor$)+$D^{\mathcal{O}(n)}$DegreeRoadmap(D, $n - \lfloor\sqrt{n}\rfloor$, $n - \lfloor\sqrt{n}\rfloor - 1$)

Total Degree: $D^{\mathcal{O}(n\sqrt{n})}$

Obtaining arithmetic complexity results:

- Hard to obtain via Gröbner bases, while it is relevant to use them in practice
- Lecerf's results on geometric resolution in non-equi-dimensional (or non-radical) situations
- Schost's results (thesis) for dealing with parameters in this context – usefull to manage infinitesimals in the theoretical complexity viewpoint

$\longrightarrow$ Probabilistic algorithm whose arithmetic complexity is $D^{\mathcal{O}(n\sqrt{n})}$

Conclusions and Perspectives

- Is this algorithm more efficient than the one of S./Mezzarrobba ? **No**.
- Can this theoretical complexity be improved? **Probably, YES!**
  - What can be expected? Our hope is $D^{\mathcal{O}(n \ln n)}$
  - Is $D^{\mathcal{O}(n)}$ reachable? The only thing we know is that we will try!
- Is there hope to obtain an efficient algorithm from these techniques? **Yes**.
  - Avoid singular fibers (but it's harder in our context)
  - Follow step by step the process of research which has lead to efficient algorithms for computing sampling points in real algebraic sets.
- Are there some other results that could be expected from this work? **Perhaps**
  - Complexity of computing roadmaps in real algebraic sets defined by s quadratic polynomials? (Thom-Porteous-Gambelli Formulas)
  - Complexity of describing semi-algebraically the connected components of real algebraic sets?